VARIATIONAL PROBLEMS OF GAS DYNAMICS OF NONEQUILIBRIUM AND EQUILIBRIUM FLOWS

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The problem of shape determination is examined for two-dimensional and axisymmetric bodies with minimum drag and for nozzles with maximum thrust under conditions of steady supersonic flow of inviscid and thermally nonconducting gas in the presence and in the absence of irreversible processes of the type of chemical reactions proceeding at finite rates. It assumed that the region of influence of the part of the contour which is to be determined is bounded by characteristics and does not contain shock waves. The boundaries with respect to the body contour are arbitrary; the dimensions of the body, the area of the surface, the volume etc. can be prescribed.

In the present study parameters on the surface of the body determined by a system of non-linear equations in partial differentials, appear as functionals of a form which is unknown in advance. In problems solved up to recent time [1 to 9] this difficulty is overcome by a transformation to a control contour as suggested by Nikol'skii [10]. However, this transformation is applicable when only the dimensions of the body are prescribed and when irreversible processes are not present.

A method for solution of problems which do not permit such a transformation was proposed recently by Guderley and Armitage [11] and independently by Sirazetdinov [12]. Application of this method to problems of the present study permits to obtain the necessary conditions of an extremum which serves as a basis for the construction of optimum contours. Furthermore, it is demonstrated that in a number of cases it is necessary to permit discontinuities in Lagrange's multipliers for continuous parameters of flow. It is shown that these discontinuities can occur along characteristics and streamlines. Relationships for discontinuities are obtained.

1. Let x and y be orthogonal coordinates; in the axisymmetric cases the x-axis is oriented along the axis of symmetry from left to right. As independent variables we take y and the flow function ψ, which is determined by Equation

$$d\psi = y^\nu \rho\, (-vdx + udy)$$

where ρ is density, u, v are projections of velocities on the x and y-axes; $\nu = 0$ and 1 for the two-dimensional and axisymmetric case respectively. With the adopted variables the flow is described by Equations

$$L_1 \equiv \frac{\partial u}{\partial y} - \frac{\partial y^\nu p}{\partial \psi} = 0 \quad \text{(of motion)} \tag{1.1}$$

$$L_2 \equiv \frac{\partial (y^\nu \rho v)^{-1}}{\partial y} + \frac{\partial (u/v)}{\partial \psi} = 0 \quad \text{(of continuity)} \tag{1.2}$$

$$L_3 \equiv \rho u \frac{\partial u}{\partial y} + \rho v \frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = 0 \quad \text{(of motion)} \tag{1.3}$$

$$L_4 \equiv \frac{\partial x}{\partial y} - \frac{u}{v} = 0 \quad \text{(of streamlines)} \tag{1.4}$$

$$\frac{u^2 + v^2}{2} + h = H(\psi) \quad \text{(of energy)} \tag{1.5}$$

Here p is the pressure, h the specific enthalpy, H total enthalpy which is a known function of ψ. Let the thermodynamic state of the gas be defined by the pressure, by the temperature T of approaching degrees of freedom of some component of gas and by n parameters q_i connected with irreversible processes ($i = 1, \ldots, n$). These parameters can be concentrations of components, energies of various degrees of freedom etc. Let us introduce an n-dimensional vector $\mathbf{q} = (q_1, \ldots, q_n)$. Functions of the type $f(q_1, \ldots, q_n)$ will be written in the form $f(\mathbf{q})$. By virtue of the above mentioned, Equation of state and Expression for h have the form

$$\rho = \rho(p, T, \mathbf{q}, \psi), \qquad h = h(p, T, \mathbf{q}, \psi) \tag{1.6}$$

The change in parameters $\mathbf{q}$ is described by Equations

$$\mathbf{L} \equiv \frac{\partial \mathbf{q}}{\partial y} - \frac{\boldsymbol{\omega}(p, T, \mathbf{q}, \psi)}{v} = 0 \tag{1.7}$$

where $\mathbf{L}$ and $\boldsymbol{\omega}$ are vectors with components L_i and ω_i ; ω_i is the rate of change of parameter q_i . The right-hand members of (1.6) and ω_i are known functions of p, T, $\mathbf{q}$ and ψ . The presence of ψ shows that different gases can flow along different streamlines.

We introduce the sound velocity c through Equation

$$c^{-2} = \rho_p + \frac{\rho_T}{h_T}\left(\frac{1}{\rho} - h_p\right) \tag{1.8}$$

Here

$$\rho_p = \left(\frac{\partial \rho}{\partial p}\right)_{T,\mathbf{q},\psi}, \qquad \rho_T = \left(\frac{\partial \rho}{\partial T}\right)_{p,\mathbf{q},\psi}, \qquad h_p = \left(\frac{\partial h}{\partial p}\right)_{T,\mathbf{q},\psi}, \qquad h_T = \left(\frac{\partial h}{\partial T}\right)_{p,\mathbf{q},\psi}$$

Equations (1.1) to (1.7) form a complete system. For $u^2 + v^2 - w^2 > c^2$ this system has three families of real characteristics. These streamlines with ψ = const for which Equations (1.3) to (1.5) are satisfied, and Mach lines for which

$$dx \mp \frac{u\sqrt{w^2 - c^2} \mp cv}{y^\nu \rho c w^2}\, d\psi = 0 \tag{1.9}$$

$$dy \mp \frac{v\sqrt{w^2 - c^2} \pm cu}{y^\nu \rho c w^2}\, d\psi = 0 \tag{1.10}$$

$$u^2 d\frac{v}{u} \pm \frac{\sqrt{w^2 - c^2}}{\rho c}\, dp + \left\{\frac{\nu v}{y} + \left(\boldsymbol{\rho} - \frac{\rho_T \mathbf{h}}{h_T}\right)\boldsymbol{\omega}\right\}\frac{d\psi}{y^\nu \rho} = 0 \qquad (1.11)$$

Here, vectors are

$$h = (h_{q_1}, \ldots, h_{q_n}), \quad \rho = (\rho_{q_1}, \ldots, \rho_{q_n})$$

$$h_{q_i} = \left(\frac{\partial h}{\partial q_i}\right)_{p,T,\psi,q_j \neq q_i}, \quad \rho_{q_i} = \left(\frac{\partial \rho}{\partial q_i}\right)_{p,T,\psi,q_j \neq q_i}$$

The upper index refers to characteristics of the first family.

All variables are dimensionless. The reduction to dimensionless form is achieved by dividing x and y by l, the velocities by w_∞, the densities by ρ_∞, pressures by $\rho_\infty w_\infty^2$, temperatures by $R^{-1} w_\infty^2$, enthalpies by w_∞^2 and ψ by $l^{(\nu+1)}\rho_\infty w_\infty$, where l, w_∞ and ρ_∞ are quantities with dimensions of length, velocity and density, R is the gas constant of certain gas. The parameters q_i are reduced to nondimensional form by taking into account their various dimensions.

2. In the problem under consideration it is required to find the necessary conditions determining the form of the contour ag which insures a minimum of wave drag (Fig. 1) or a maximum of thrust (Fig. 2) for a given flow to the left of the characteristic ac,

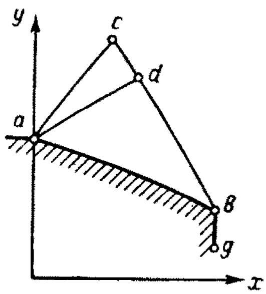

Fig. 1

In addition to the position of point a, the length of the body X, the area of the side surface, the volume etc. can be prescribed.

The desired contour may consist of regions of two-sided and outer extremums. The regions are determined by the statement of the problem and the limits of applicability of utilized equations. For fixed length this will be the section bg of the straight line $x = X$, where $\delta x \leqslant 0$ is permissible. For a given cross-sectional dimension Y the region of the outer extremum will be a section of the straight line $y = Y$. In the axisymmetric case and in the case of a symmetrical flat body such a region coincides with a section of the axis of symmetry. To the limit of applicability of Equations (1.1) to (1.7) without taking into account shock wave relations, there corresponds a section of the curve of maximum compression [6] such that the shock wave which is formed by the approaching flow starts at the boundary of the region of influence for the desired area of the body.

The direction of the prescribed contour to the left of a and the direction of the contour which is determined from the solution of the variational problem are different in the general case. At a, therefore, passage of flow over a convex corner takes place (the case where the flow over the contour ab results in shock wave formation at a is not considered here). On the basis of technical consideration a discontinuity in the vicinity of a can be forbidden, for example, due to boundaries on $x'' = d^2x/dy^2$. This will also give some part of the outer extremum.

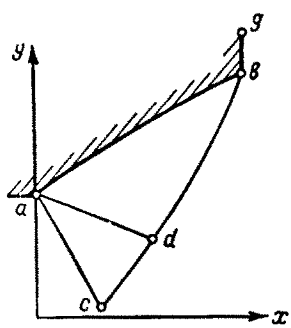

Fig. 2

Among the enumerated regions of the outer extremum we will examine only the first and the last and we will investigate only the case where the parameters are constant in the vicinity of bg and do not depend on the shape of the contour ag. With an accuracy to an insignificant factor the wave drag or thrust are given by

$$\chi = \int_a^b y^\nu p\, dy + \int_b^g y^\nu p_T\, dy$$

where b is the connecting point of the region with a two-sided extremum and the section bg, p_T = const is pressure on the bg; points b and g may coincide.

We have isoperimetric conditions (taking into consideration that $u = vx'$) in the form

$$K^j = \int_a^b f^j (y, x, v, p, T, \mathbf{q}, x')\, dy + \int_b^g f^{0j} (y, x, x')\, dy \qquad (j = 1, \ldots, m) \qquad (2.1)$$

where K^j are given constants, f^j and f^{0j} are known functions, m is number of isoperimetric conditions; prime designates derivatives $(\partial/\partial y)_{\psi=\psi_a}$; indices $a, b, \ldots$ are given to parameters at corresponding points.

3. We utilize the methods of Guderley, Armitage and Sirazetdinov [11 and 12]. On the surface of the body the flow parameters are determined by Equations (1.5) and (1.6) and by differential Equations (1.3), (1.4) and (1.7) along ab and (1.1) to (1.3) and (1.7) in the region of influence G, bounded by characteristics ac and cb and by contour ab. We construct the functionals

$$I = \int_a^b (\Phi + \alpha L_3 + \beta L_4 + \boldsymbol{\gamma} \mathbf{L})\, dy + \int_b^g F dy + \iint_G (\mu_1 L_1 + \mu_2 L_2 + \mu_3 L_3 + \mathbf{Q}\mathbf{L})\, dy\, d\psi$$

$$\Phi = \Phi (y, x, v, p, T, \mathbf{q}, x', \lambda) = y^\nu p + \sum_{j=1}^m \lambda^j f^j (y, x, v, p, T, \mathbf{q}, x')$$

$$F = F (y, x, x', \lambda) = y^\nu p_T + \sum_{j=1}^m \lambda^j f^{0j} (y, x, x')$$

Here $\lambda^1, \ldots, \lambda^m$ are constants, $\alpha (y)$, $\beta (y)$, $\gamma (y)$, $\mu_1 (y, \psi)$, $\mu_2 (y, \psi)$, $\mu_3 (y, \psi)$, $\mathbf{Q} (y, \psi)$ are variable Lagranges multipliers, $\boldsymbol{\gamma}$ and $\mathbf{Q}$ are n-dimensional vectors. By virtue of Equations (1.1) to (1.4) and (1.7) and conditions (2.1), variations of I and χ coincide for permissible variations.

We shall find the first variation of I in the absence of limitations with respect to x'', i.e. for the case where a discontinuity in the contour is permitted (Fig. 1 and 2).

By virtue of (1.5) and (1.6)

$$\delta T = -\frac{u}{h_T}\,\delta u - \frac{v}{h_T}\,\delta v - \frac{h_p}{h_T}\,\delta p - \frac{\mathbf{h}}{h_T}\,\delta \mathbf{q}$$

Therefore if $\zeta = \zeta(p, T, \mathbf{q}, \psi)$, then

$$\delta\zeta = -\frac{\zeta_T u}{h_T}\,\delta u - \frac{\zeta_T v}{h_T}\,\delta v + \left(\zeta_p - \frac{\zeta_T h_p}{h_T}\right)\delta p + \left(\boldsymbol{\zeta} - \frac{\zeta_T \mathbf{h}}{h_T}\right)\delta \mathbf{q}$$

Here

$$\boldsymbol{\zeta} \equiv \zeta_{\mathbf{q}} = (\zeta_{q_1}, \ldots, \zeta_{q_n})$$

$$\zeta_p = \left(\frac{\partial\zeta}{\partial p}\right)_{T,\mathbf{q},\psi} \qquad \zeta_T = \left(\frac{\partial\zeta}{\partial T}\right)_{p,\mathbf{q},\psi}, \qquad \zeta_{q_i} = \left(\frac{\partial\zeta}{\partial q_i}\right)_{p,T,q_j\neq q_i,\psi}$$

In order to eliminate variations under the integral signs which appear in the process of variation due to derivatives, we utilize the formula for integration by parts and a relationship which is a consequence of Green's formula

$$\iint\limits_G \left(N\frac{\partial\delta\xi}{\partial y} + M\frac{\partial\delta\xi}{\partial\psi}\right)dy\,d\psi = -\iint\limits_G \left(\frac{\partial N}{\partial y} + \frac{\partial M}{\partial\psi}\right)\delta\xi\,dy\,d\psi + \oint\left(M\frac{d\psi}{dy} - N\right)\delta\xi\,dy$$

where the contour integral is taken along the unvaried boundary of the region of influence in the plane $y\psi$. Variation of double integral, which is related to a change in boundary G, is not present because the expression under the integral is equal to zero. However, variations of integral along ag, which are connected with a change of coordinates b and g are different from zero. Increments in coordinates of these points will be designated by Δy and Δx. With an accuracy to small terms of higher order $\delta x = \Delta x + x'\Delta y$.

Let ad be the closing characteristic of an expansion wave fan. Small changes of the contour ag have no influence on the flow in acd. Therefore here, including ac and cd, variations of parameters are equal to zero. Further, δx_a disappears because a is given, the variation $\delta\mathbf{q}_a$ disappears by virtue of Equations (1.7); finally, δu_a, δv_a and δp_a are connected with the equality $(\rho u\delta u + \rho v\delta v + \delta p)_a = 0$. The latter follows from Equation (1.3), which at a has the form $\rho u du + \rho v dv + dp = 0$ and the fact that here $u = u(p)$ and $v = v(p)$ and consequently $\delta u = (du/dp)\delta p$ and $dv = (dv/dp)\delta p$.

By taking into account what was mentioned above and by utilizing (1.3), (1.7) and (1.8) we find

$$\begin{aligned}\delta\chi = \delta I = {} & \{\Phi_- - F_+ - (\Phi_{x'} + \beta)_- x_-' + F_{x'_+}x_+'\}_b\,\Delta y_b + (\Phi_{x'_-} + \beta_- - F_{x'_+})_b\,\Delta x_b + \\ & + \{\alpha(\rho u\delta u + \rho v\,\delta v + \delta p) + \boldsymbol{\gamma}\delta\mathbf{q}\}_b + F_g\Delta y_g + \\ & + \int\limits_a^b (U^\circ\delta x + U^1\,\delta u + U^2\delta v + U^3\delta p + \mathbf{U}\delta\mathbf{q})\,dy + \int\limits_b^g \{F_x - (F_{x'})'\}\,\delta x\,dy + \\ & + \int\limits_b^d (V^1\,\delta u + V^2\,\delta v + V^3\,\delta p + \mathbf{V}\delta\mathbf{q})\,dy + \\ & + \iint\limits_{G^\circ} (W^1\,\delta u + W^2\,\delta v + W^3\,\delta p + \mathbf{W}\delta\mathbf{q})\,dy\,d\psi \end{aligned} \tag{3.1}$$

$$(\Phi_{x'} = (\partial\Phi/\partial x')_{y,\,x,\,v,\,p,\,T,\,\mathbf{q},\,\lambda}, \qquad F_{x'} = (\partial F/\partial x')_{y,\,x,\,\lambda})$$

where G° is the region adb; minus and plus subscripts are attached to parameters at point b before and after the discontinuity. U^i, V^i, W^i, $\mathbf{U}$,

V and W are known functions of flow parameters and Lagrange's multipliers.

4. Let us examine various terms of Expression (3.1). For any contour ag some of them can be reduced to zero by a special choice of Lagrange's multipliers. We will determine μ_1, μ_2, μ_3. $\mathbf{Q}$ is obtained from Equations $W^1 = 0$, $W^2 = 0$, $W^3 = 0$, $\mathbf{W} = 0$, which we will represent in the form

$$W^2 - \frac{v}{u} W^1 \equiv \frac{v}{u}\frac{\partial \mu_1}{\partial y} + \frac{1}{y^\nu \rho v^2}\frac{\partial \mu_2}{\partial y} + \frac{w^2}{uv^2}\frac{\partial \mu_2}{\partial \psi} + \frac{Q\omega}{v^2} = 0 \qquad (4.1)$$

$$W^2 - \rho v W^3 \equiv -y^\nu \rho v \frac{\partial \mu_1}{\partial \psi} + \frac{c^2 - v^2}{y^\nu \rho v^2 c^2}\frac{\partial \mu_2}{\partial y} + \frac{u}{v^2}\frac{\partial \mu_2}{\partial \psi} - \mu_3 \left(\rho - \frac{\rho_T \mathbf{h}}{h_T}\right)\omega + $$
$$+ \mathbf{Q}\left\{\rho\left(\omega_p - \frac{\omega_T h_p}{h_T}\right) + \frac{\omega}{v^2} + \frac{\omega_T}{h_T}\right\} = 0 \qquad (4.2)$$

$$-W^1 - \frac{v}{u} W^2 \equiv \frac{\partial \mu_1}{\partial y} - \frac{1}{y^\nu uv}\left(\frac{1}{\rho} - \varepsilon w^2\right)\frac{\partial \mu_2}{\partial y} + \frac{\rho w^2}{u}\frac{\partial \mu_3}{\partial y} + \qquad \left(\varepsilon = \frac{\rho_T}{\rho^2 h_T}\right)$$
$$+ \frac{\mu_3 \rho w^2}{u}\left\{\left(\frac{1}{\rho c^2} - \varepsilon\right)\frac{\partial p}{\partial y} + \left(\rho - \frac{\rho_T \mathbf{h}}{h_T}\right)\frac{\omega}{\rho v}\right\} - \frac{\mathbf{Q}}{uv}\left(\omega + \frac{\omega_T w^2}{h_T}\right) = 0 \qquad (4.3)$$

$$\mathbf{W} \equiv \frac{1}{y^\nu \rho^2 v}\left(\rho - \frac{\rho_T \mathbf{h}}{h_T}\right)\frac{\partial \mu_2}{\partial y} - \frac{\partial \mathbf{Q}}{\partial y} - \frac{\mu_3}{\rho}\left(\rho - \frac{\rho_T \mathbf{h}}{h_T}\right)\frac{\partial p}{\partial y} + $$
$$+ \frac{(\mathbf{Q}\cdot\omega_T)\,\mathbf{h}}{v h_T} - (\mathbf{Q}^\bullet\cdot \omega)_{\mathbf{q}}\, v^{-1} = 0 \qquad (4.4)$$

The system of Equations (4.1) to (4.4) is of the same type as the system of flow equations. It is elliptical for $w < c$ and hyperbolic for $w > c$. For $w > c$ there are three families of real characteristics, which coincide with characteristics of Equations (1.1) to (1.7). Along the streamline Equations (4.3) and (4.4) are satisfied, along Mach lines

$$d\mu_1 \pm \frac{\sqrt{w^2 - c^2}}{y^\nu \rho v^2 c} d\mu_2 + \left\{\mu_3 \rho \omega \left(\rho - \frac{\rho_T \mathbf{h}}{h_T}\right) - \mathbf{Q}\omega_T h_T^{-1} + \right.$$
$$\left. + \mathbf{Q}\left(\omega_p - \frac{\omega_T h_p}{h_T}\right)\rho\right\}\frac{d\psi}{y^\nu \rho v} + \frac{Q\omega}{v^2}dx = 0 \qquad (4.5)$$

where the upper sign corresponds to characteristics of the first family.

We obtain boundary conditions along the characteristic db and the contour ab.

By equating coefficients in front of δu, δv and $\delta \mathbf{q}$ along db to 0 we obtain

$$\mu_1 \pm \frac{\sqrt{w^2 - c^2}}{y^\nu \rho v^2 c}\mu_2 = 0 \qquad (4.6)$$

$$\mu_1 + \mu_2\left(\varepsilon u y^{-\nu} - \frac{dy}{d\psi}\right)v^{-1} + \mu_3 \rho u = 0 \qquad (4.7)$$

$$\mu_2\left(\rho - \frac{\rho_T \mathbf{h}}{h_T}\right) - \mathbf{Q} y^\nu \rho^2 v = 0 \qquad (4.8)$$

If not especially mentioned, here and below the upper signs refer to the external problem (Fig.1) and the lower signs to the internal problem (Fig.2); $dy/d\psi$ along db is determined from (1.10). Satisfaction of (4.6) and (4.7) along db leads also to a transformation to zero for the coefficient of V^3 by virtue of (1.10). We find the boundary condition for ab by examining coefficients of U^1, U^2, U^3, $\mathbf{U}$. Along ab we determine multipliers α, β, γ and μ_1, from Equations $U^1 = 0$, $U^2 = 0$, $U^3 = 0$, $\mathbf{U} = 0$, which by taking into consideration equations for U^1 and $\mathbf{U}$ can be presented in the form

$$-\mathbf{U} \equiv \gamma' - \Phi_q + \frac{\Phi_T \mathbf{h}}{h_T} + \frac{\alpha}{\rho}\left(\boldsymbol{\rho} - \frac{\rho_T \mathbf{h}}{h_T}\right)p' + \frac{1}{v}\,(\gamma\cdot\boldsymbol{\omega})_{\mathbf{q}} - \frac{(\gamma\cdot\boldsymbol{\omega}_T)\,\mathbf{h}}{h_T v} = 0$$

$$-\frac{u}{v}U^1 - U^2 \equiv \frac{w^2}{v}(\alpha\rho)' - \Phi_v + \tag{4.9}$$

$$+\frac{w^2}{h_T v}(\Phi_T - \alpha\rho_T\rho^{-1}p' - \gamma\omega_T v^{-1}) - \gamma\omega v^{-2} = 0 \tag{4.10}$$

$$U^3 \equiv \pm y^\nu \mu_1 + \Phi_p - \frac{v\Phi_v}{\rho w^2} + \frac{\Phi_T}{h_T}\left(\frac{1}{\rho} - h_p\right) + \frac{\alpha\omega}{\rho v}\left(\boldsymbol{\rho} - \frac{\rho_T \mathbf{h}}{h_T}\right) -$$

$$-\frac{\gamma}{v}\left\{\frac{\omega}{\rho w^2} + \omega_p + \frac{\omega_T}{h_T}\left(\frac{1}{\rho} - h_p\right)\right\} = 0 \tag{4.11}$$

$$uU^2 - vU^1 \equiv u\Phi_v + \gamma\omega u v^{-2} + \frac{w^2}{v^2}(\beta \pm \mu_2) = 0 \tag{4.12}$$

$$\Phi_v = (\partial\Phi/\partial v)_{y,\,x,\,p,\,T,\,\mathbf{q},\,x',\,\lambda}, \qquad \Phi_p = (\partial\Phi/\partial p)_{y,\,x,\,v,\,T,\,\mathbf{q},\,x',\,\lambda}$$

$$\Phi_T = (\partial\Phi/\partial T)_{y,\,x,\,v,\,p,\,\mathbf{q},\,x',\,\lambda}, \qquad \Phi_{q_i} = (\partial\Phi/\partial q_i)_{y,\,x,\,v,\,p,\,T,\,q_j\neq q_i,\,x',\,\lambda}$$

Here U^3 is transformed by taking into account (4.10). As initial conditions for integration of (4.9) and (4.10) we take

$$\alpha_b = 0, \qquad \gamma = 0 \tag{4.13}$$

Therefore, for any smooth contour ab , multipliers α, β, γ, μ_1, μ_2, μ_3 and $\mathbf{Q}$ can be selected such that coefficients of U^i, V^i, W^i $(i = 1,2,3)$ $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$, α_b, γ_b in Equation (3.1) are transformed to 0 . Actually, it is required for this that relationships (4.1) to (4.13) are fulfilled. For any contour ab , the flow in acb can be calculated, for example, by the method of characteristics and consequently is known. For known flow parameters, α and γ along ab are determined by Equations (4.9) and (4.10) and conditions (4.13). Subsequently μ_1, in particular μ_{1b}, is found along ab from (4.11), and from (4.5) to (4.8) with utilization of μ_{1b}, the values μ_1, μ_2, μ_3 and $\mathbf{Q}$ are determined along db . Values of these quantities along the characteristic db , and μ_1 along the contour ab with the aid of Equations (4.1) to (4.4) or of equivalent Equations (4.3) to (4.5) permit to find μ_1, μ_2, μ_3, and $\mathbf{Q}$ in the region G°. Finally, the multiplier β along ab is determined by Equation (4.12). It is clear that the Lagrange multipliers found in this fashion are dependent on the shape of the contour ab .

5. If ab contains a discontinuity (Fig.3), then it is not possible to satisfy all obtained conditions with Lagrange's multipliers which are continuous in G . Actually, μ_1, μ_2, μ_3, and $\mathbf{Q}$ along characteristic ke are found from conditions along kb and db. Found in this fashion, the value μ_{1k} to the left of the discontinuity will not satisfy Equation (4.11) in the general case. Consequently, it is necessary to admit the possibility of lines of discontinuity in Lagrange's multipliers for continuous parameters of flow.

Let l be such a line. In the variation of I the region G is divided into regions of continuous Lagrange's multipliers. In these regions and along the boundaries ab and db the functions μ_1, μ_2, μ_3 and $\mathbf{Q}$ are determined in the previous fashion, i.e. Equations (4.1) to (4.13) are satisfied. Let $[\varphi]$ be a jump φ along l. Since the flow parameters and their variations are continuous along l, there appears an additional integral in Expression (3.1)

$$\int\limits_l \left(S^1\delta u + S^2\delta v + S^3\delta p + \mathbf{S}\ \frac{d\psi}{dy}\,\delta\mathbf{q}\right) dy$$

where S^1, S^2, S^3, $\mathbf{S}$ are linear orthogonal functions of $[\mu_1]$, $[\mu_2]$, $[\mu_3]$ and $[\mathbf{Q}]$, which also depend on flow parameters and $d\psi/dy$ along l. We will determine $[\mu_1]$, $[\mu_2]$, $[\mu_3]$ and $[\mathbf{Q}]$ such that the following conditions are fulfilled along l

$$S^1 = 0, \quad S^2 = 0, \quad S^3 = 0, \quad \mathbf{S}\,\frac{d\psi}{dy} = 0 \tag{5.1}$$

If l is not a streamline and not a characteristic, this gives $(n + 3)$ linearly independent linear homogeneous equations with respect to $(n + 3)$ variables $[\mu_1]$, $[\mu_2]$, $[\mu_3]$, $[\mathbf{Q}]$. Consequently, in this case we have

$$[\mu_1] = [\mu_2] = [\mu_3] = 0, \qquad [\mathbf{Q}] = 0$$

i.e. the discontinuity is not present.

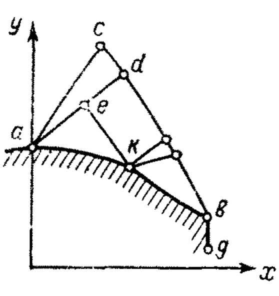

Fig. 3

If l is a characteristic, then S^3 is a linear combination of S^1 and S^2 and conditions (5.1) give

$$[\mu_1] \mp [\mu_2]\,\frac{\sqrt{w^2 - c^2}}{y^\nu \rho v^2 c} = 0 \tag{5.2}$$

$$[\mu_1] + [\mu_2]\left(\varepsilon u y^{-\nu} - \frac{dy}{d\psi}\right) v^{-1} + [\mu_3]\,\rho u = 0 \tag{5.3}$$

$$[\mu_2]\left(\boldsymbol{\rho} - \frac{\rho_T \mathbf{h}}{h_T}\right) - [\mathbf{Q}]\, y^\nu \rho^2 v = 0 \tag{5.4}$$

Furthermore, since (4.5) is satisfied from each side of the characteristic, then

$$d\,[\mu_1] \pm \frac{\sqrt{w^2 - c^2}}{y^\nu \rho v^2 c}\, d\,[\mu_2] + \left\{ [\mu_3]\,\rho\boldsymbol{\omega}\left(\boldsymbol{\rho} - \frac{\rho_T \mathbf{h}}{h_T}\right) - [\mathbf{Q}]\,\boldsymbol{\omega}_T h_T^{-1} - \right.$$
$$\left. - [\mathbf{Q}]\left(\boldsymbol{\omega}_p - \frac{\boldsymbol{\omega}_T h_p}{h_T}\right)\rho\right\}\frac{d\psi}{y^\nu \rho v} + [\mathbf{Q}]\,\boldsymbol{\omega} v^{-2}\, dx = 0 \tag{5.5}$$

Here and in (5.2) the upper sign corresponds to a characteristic of the first family; $dy/d\psi$ in (5.3) is determined from (1.10). Equations (5.2) to (5.5) determine the jump in all quantities along a given characteristic from the jump in one of these quantities at some point. By virtue of linearity and homogeneity of (5.2) to (5.5), all Lagrange multipliers are either continuous or discontinuous along the entire characteristic.

If l is a streamline, then $d\psi/dy = 0$ and in addition to this S^3 is a linear combination of S^1 and S^2. As a result we obtain

$$[\mu_1] = [\mu_2] = 0 \tag{5.6}$$

From this condition and Equations (4.3) and (4.4)

$$\frac{\partial [\mu_3]}{\partial y} + [\mu_3] \left\{ \left(\frac{1}{\rho c^2} - \varepsilon \right) \frac{\partial p}{\partial y} + \left(\rho - \frac{\rho_T \mathbf{h}}{h_T} \right) \frac{\boldsymbol{\omega}}{\rho v} \right\} - \frac{[\mathbf{Q}]}{\rho v} \left(\frac{\boldsymbol{\omega}}{w^2} + \frac{\boldsymbol{\omega}_T}{h_T} \right) = 0 \tag{5.7}$$

$$\frac{\partial [\mathbf{Q}]}{\partial y} + \frac{[\mu_3]}{\rho} \left(\rho - \frac{\rho_T \mathbf{h}}{h_T} \right) \frac{\partial p}{\partial y} - ([\mathbf{Q}] \cdot \boldsymbol{\omega}_T) \frac{\mathbf{h}}{v h_T} + ([\mathbf{Q}] \cdot \boldsymbol{\omega})_{\mathbf{q}} v^{-1} = 0 \tag{5.8}$$

These equations are also linear and homogeneous, consequently, if only at one point of the streamline, $[\mu_3] = 0$ and $[\mathbf{Q}] = 0$, then these conditions are fulfilled along the entire streamline.

Thus the introduction of discontinuities permits to satisfy all conditions of the previous section. In particular, in the case shown in Fig.3 the line of discontinuity will be the characteristic ke.

Continuity in flow parameters was assumed above. Discontinuities in flow parameters, for example shock waves, will also be discontinuities in Lagrange's multipliers. When relationships are obtained at such discontinuities, it is necessary to take into account the relationship between the variation of flow parameters from different sides of the discontinuity.

6. In accordance with the choice of Lagrange's multipliers the expression for $\delta\chi$ becomes

$$\delta\chi = \delta I = [\Phi_- - F_+ - (\Phi_{x'} + \beta)_- x_-']_b \Delta y_b + (\Phi_{x'_-} + \beta_- - F_{x'_+})_b \Delta x_b + \\ + F_g \Delta y_g + \int_a^b U^\circ \delta x \, dy + \int_b^g \{F_x - (F_{x'})'\} \delta x \, dy \tag{6.1}$$

where, in contrast to (3.1), all variations are independent.

In the region of a two-sided extremum ab the variations in x are arbitrary, consequently the necessary condition for an extremum has the form

$$U^\circ \equiv \Phi_x - (\beta' + \Phi_{x'}') = 0 \qquad (\Phi_x = (\partial\Phi / \partial x)_{y,\, v,\, p,\, T,\, \mathbf{q},\, x',\, \lambda}) \tag{6.2}$$

For an arbitrary length there is no end, and in Expression (6.1) only two first terms remain, and these are without F_+ and $F_{x'+}$. Since Δx_b is arbitrary, the length of the contour is determined by condition

$$(\Phi_{x'} + \beta)_{b_-} = 0 \tag{6.3}$$

while the ordinate $y_b = y_s$ is either given or is found from condition

$$\{\Phi - (\Phi_{x'} + \beta)\, x'\}_{b_-} = 0 \tag{6.4}$$

For a limited length the end may also be absent. The ordinate y_b as before is either given or determined from (6.4). In the first case $\Delta y_b \gtrless 0$ are admissible (the upper sign refers to the external problem) and the necessary condition for a minimum of drag or a minimum of thrust will be

$$\{\Phi_- - F_+ - (\Phi_{x'} + \beta)_- \; x_-'\}_b \geqslant 0 \tag{6.5}$$

If an end is present, y_b is found from condition (6.5) with the equal sign. Then the transverse dimension is undefined, the ordinate y_s is determined from

$$F_g = 0 \tag{6.6}$$

Furthermore, in this case Δx_b and δx are negative along bg. Consequently, the necessary conditions for an outer extremum will be

$$(\Phi_{x'_-} + \beta_- - F_{x'_+})_b \lesseqgtr 0 \tag{6.7}$$

$$F_x - (F_{x'})' \geqslant 0 \quad \textbf{along } bg \tag{6.8}$$

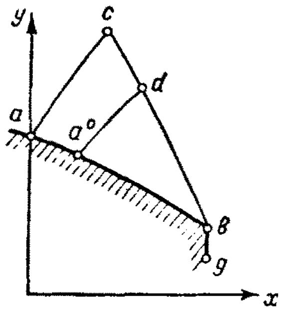

Fig. 4

The equations obtained constitute a system of necessary conditions which determine the shape of the optimum contour. Freedom of the choice of characteristic ad permits the construction of a contour of the required length. The selection of Lagrange's constant multipliers satisfies conditions (2.1).

Equations (4.6) to (4.8), (4.11) and (4.12) written at point b, and conditions (4.13) allow to express μ_{1b}, μ_{2b}, μ_{3b}, Q_b' and β_b through flow parameters y and x at b. In particular

$$\beta_b = \left(\mp \frac{\rho v^2 c}{\sqrt{w^2 - c^2}} \left\{\Phi_p - \frac{v\Phi_v}{\rho w^2} + \frac{\Phi_T}{h_T}\left(\frac{1}{\rho} - h_p\right)\right\} - \frac{u v^2}{w^2}\Phi_v\right)_b$$

Substitutions of this expression into (6.3) to (6.5) and (6.7) leads to relationships which for the optimum contour must be satisfied in b by flow parameters y and x. For example, in the absence of isoperimetric conditions, (6.5) yields the Busemann condition [1]. The presence of irreversible processes is reflected on these relationships through the form of derivatives with respect to p and T. The same relationships can be obtained in a different way if one takes into account that for the optimum contour the end element of the contour ab and the end are also optimum.

7. If in the vicinity of a the following limitation is imposed on x''

$$|x''| \leqslant K(y) \tag{7.1}$$

where $K(y)$ is a given function, then instead of a discontinuity in a, there is a region of outside extremum $aa°$

$$x'' = \pm K(y) \ \operatorname{sign} x \tag{7.2}$$

which smoothly joins with the region of two-sided extremum $a^\circ b$. Now (Fig. 4 and 5) the variations of parameters can be different from zero in the entire region G. Therefore we will require that Equations (4.1) to (4.5) be satisfied also in the entire region G, and Equations (4.6) to (4.8) be satisfied along the entire characteristic ob. However, the necessary condition for two-sided extremum (6.2) is now satisfied only along $a^\circ b$. Other relationships are satisfied without change.

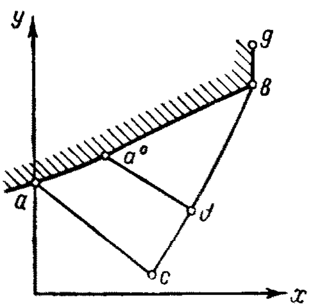

Fig. 5

As a result we obtain

$$\delta\chi = \int_a^{a^\circ} U^\circ \delta x \, dy$$

Let us vary x'' only in the region mn to the right of the point m on aa°, and let $\max|\delta x''|$ and $|y_m - y_n|$ be small of the same order of magnitude. With an accuracy to quantities of higher order

$$\delta\chi = \left(\int_m^{a^\circ} (y - y_m) \, U^\circ \, dy\right) \int_m^n \delta x'' \, dy$$

According to (7.1) and (7.2) for admissible $\delta x''$

$$\int_m^n \delta x'' \, dy \leqslant 0$$

Consequently, the condition that aa° is a region of outer extremum has the form

$$\int_m^{a^\circ} (y - y_m) \, U^\circ \, dy \leqslant 0 \tag{7.3}$$

for any point m along aa°. We note that a sufficient condition for fulfillment of this inequality will be

$$U^\circ \equiv \Phi_x - (\Phi_{x'} + \beta)' \leqslant 0 \qquad \text{along } aa^\circ \tag{7.4}$$

8. Conditions determining the optimum contour for equilibrium and frozen flows are obtained from relationships found above by taking into account that in these cases parameters $\mathbf{q}$ which vary according to Equations (1.7), are absent. Therefore, in order to obtain the mentioned conditions it is sufficient to omit Equations (4.7) and (4.9) and terms which contain $\boldsymbol{\rho}$, $\mathbf{h}$, $\boldsymbol{\omega}$, $\mathbf{Q}$ and $\boldsymbol{\gamma}$ in the other equations. Furthermore, the necessity for equations containing μ_3 and α disappears because it turns out that in this case μ_1, μ_2, β and the shape of the contour are independent of μ_3 and α.

For the solution of the problem now the following equations and conditions are used: Equation (1.3), (1.5) and (1.7) to (1.10)

$$u^2\, d\frac{v}{u} \pm \frac{\sqrt{w^2-c^2}}{\rho c}\, dp + \frac{\nu v}{y^{\nu+1}\rho}\, d\psi = 0 \tag{8.1}$$

$$d\mu_1 \pm \frac{\sqrt{w^2-c^2}}{y^{\nu}\rho v^2 c}\, d\mu_2 = 0 \tag{8.2}$$

where the upper sign refers to characteristics of the first family in the flow field; condition (4.6) applies along the closing characteristic and Equations

$$\pm\, y^{\nu}\mu_1 + \Phi_p - \frac{v\Phi_v}{\rho w^2} + \frac{\Phi_T}{h_T}\left(\frac{1}{\rho} - h_p\right) = 0 \tag{8.3}$$

$$u\Phi_v + \frac{w^2}{v^2}(\beta \pm \mu_2) = 0 \tag{8.4}$$

and (1.4) apply along ab . Equations (6.2) in the region of two-sided extremum, (6.3) to (6.5) and (6.7) in point b , (6.6) in point g , (6.8) along bg , and (7.2) to (7.4) along aa°, remain unchanged. Furthermore,

$$\rho = \rho\,(p,\, T,\, \psi), \qquad h = h\,(p,\, T,\, \psi)$$

The conditions at discontinuities can also be obtained without difficulty.

Further simplifications depend on the form of isoperimetric conditions. If p, T and v do not appear in them, then along ab

$$\mu_1 = \mp 1, \qquad \mu_2 = \mp \beta \tag{8.5}$$

In the absence of isoperimetric conditions we obtain from (6.2)

$$\beta = \beta_b = \mathrm{const} \tag{8.6}$$

Since in this case [6] all streamlines in adb or in $a^{\circ}db$ are extremals, then Equations (8.5) and (8.6) are satisfied everywhere in adb or $a^{\circ}db$. From this it is easy to find solutions which were obtained by going to a control contour.

For two-dimensional flow the problem is substantially simplified if the parameters are constant along ac . Since in this case all characteristics of the same family have the same properties as ac , from (4.6) and (8.2) to (8.4) in the region of two-sided extremum we obtain

$$\Phi_x - \left(\Phi_{x'} - \frac{uv^2}{w^2}\Phi_v \mp \left\{\Phi_p - \frac{v\Phi_v}{\rho w^2} + \frac{\Phi_T}{h_T}\left(\frac{1}{\rho} - h_p\right)\right\}\frac{\rho c v^2}{\sqrt{w^2-c^2}}\right) = 0 \tag{8.7}$$

This result is also obtained by conventional methods of variational calculus since for a given flow, parameters along ab depend only on x'. It is evident from (8.7) that if Φ is independent of x and y , the region of two-sided extremum is linear.

Derived conditions constitute a basis for construction of optimum contours with application of numerical methods. For application and verification these methods the simple solutions presented above can be used.

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